Due Sun

4.9 – Rank, Nullity, and the Fundamental Matrix Spaces

vector space spanned by the row vectors (column)

$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ 3×6

Theorem 4.9.1 The row space and column space of a matrix A have the same dimension.

same # of basis vectors, based on pivots

\# of pivots

Definition: The common dimension of the row space and column space of a matrix A is called the **rank** of A and is denoted by rank(A); the dimension of the null space of A is called the **nullity** of A and is denoted by nullity(A). That is, rank(A) = dim [row (A)] = dim [col (A)] and nullity(A) = dim [null (A)].

set of solutions to homog. system

a.k.a. dim of kernel of T_A

#1 Find the rank and nullity of the matrix A by reducing it to row echelon form.

a. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

\# leading vars: 1
\# free vars: 3
total # vars: 4

rank(A) = 1 (# pivots)
nullity(A) = 3 (# parameters)

Solution: $x_1 = -2x_2 + x_3 - x_4$

$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$

Basis has 3 vectors

b. $A = \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & -1 & -2 & -6 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

rank(A) = 3 (# pivots)
nullity(A) = 2 (# parameters)

Theorem 4.9.2 Dimension Theorem for Matrices
If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$.

total # of variables → n; # leading variables (pivots) → rank(A); # free variables (parameters) → nullity(A)

Theorem 4.9.3 If A is an $m \times n$ matrix, then
a) $\text{rank}(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.
b) $\text{nullity}(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem 4.9.4 If $A\mathbf{x} = \mathbf{b}$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters.

Rearrangement of the Dimension Theorem for Matrices.

Fundamental spaces of a matrix

Considering a matrix A and its transpose A^T, we have

$\text{row}(A)$ — same — $\text{col}(A^T)$

$\text{col}(A)$ — same — $\text{row}(A^T)$

$\text{null}(A)$ $\quad$ $\text{null}(A^T)$

The four **fundamental spaces** of a matrix A are row (A), col (A), null (A), and null $\left(A^T\right)$. The null space of A^T is also called the **left null space of A**. The **left nullity of A** is nullity $\left(A^T\right) = \dim\left[\text{null } \left(A^T\right)\right]$.

Because $A^T\vec{x} = \vec{0}$ becomes $\vec{x}^T A = \vec{0}^T$

Theorem 4.9.5 If A is any matrix, then rank(A) = rank $\left(A^T\right)$.
(# pivots doesn't change)

Let A be an $m \times n$ matrix.

Since $\text{rank}(A^T) + \text{nullity}(A^T) = m$

Thm 4.9.5 yields $\text{rank}(A) + \text{nullity}(A^T) = m$

so

$\dim[\text{row}(A)] = r \qquad \dim[\text{col}(A)] = r$

$\dim[\text{null}(A)] = n - r \qquad \dim[\text{null}(A^T)] = m - r$

To find bases of the 4 fundamental spaces of an $m \times n$ matrix A, form

$$[A \mid I_m] \xrightarrow{\text{rref}} [R \mid E]$$

- A basis for row(A) is the r rows of R that contain leading 1s (pivot rows)

- A basis for col(A) is the r columns of A that correspond to pivot columns of R

- A basis for null(A) is found in the general solution of $A\vec{x} = \vec{0}$.

- A basis for null(A^T) – a.k.a. basis for left null space — is the bottom $m-r$ rows of E.

(Pf is at the end of the section)

#11 Find the dimensions and bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix} \quad \left[\begin{array}{cc|ccc} 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ -9 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -4/3 & 0 \\ 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 9 & -12 & 1 \end{array}\right]$$

row(A) basis: $\{[1\ 0], [0\ 1]\}$

col(A) basis: $\left\{\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\right\}$

$n=2$, $r=2 \Rightarrow \text{nullity}(A) = 2-2=0$
$= \dim[\text{null}(A)]$

null(A) basis: $\emptyset = \{\ \}$

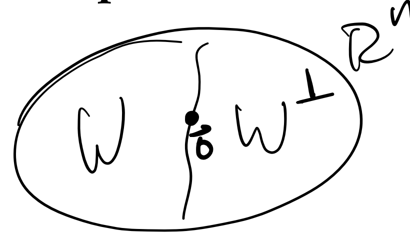
null(A^T): 3 rows (m) rank is 2 (r)
use bottom $3-2=1$ row of E

null(A^T) basis: $\{[9\ -12\ 1]\}$

Recall (Thm 3.4.3, which we saw in 3.3) that if A is an $m \times n$ matrix, then the solution set of the homogeneous system $A\vec{x}=\vec{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A. The set of solutions $\{\vec{x} \mid A\vec{x}=\vec{0}\}$ is the orthogonal complement of row(A).

$A = \{3, 5, 7, 2, 8, 6\}$ $B = \{3, 5, 7\} \subset A$ The complement of B is $\{2, 8, 6\}$

Definition: If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the **orthogonal complement** of W and is denoted by the symbol $W^{\perp}$ (pronounced "W perp").



Theorem 4.9.6 If W is a subspace of R^n, then:

a) $W^{\perp}$ is a subspace of R^n.

b) The only vector common to W and $W^{\perp}$ is $\mathbf{0}$, that is, $W \cap W^{\perp} = \{\mathbf{0}\}$.

c) The orthogonal complement of $W^{\perp}$ is W, that is, $(W^{\perp})^{\perp} = W$.

pf (a): Let $\vec{w} \in W$. $\vec{0} \perp \vec{v}$ for all $\vec{v} \in R^n$ so

$\vec{0} \perp \vec{w} \Rightarrow \vec{0} \in W^{\perp}$.

Let $\vec{u}, \vec{v} \in W^{\perp}$. Then $\vec{u} \perp \vec{w}$, $\vec{v} \perp \vec{w}$.

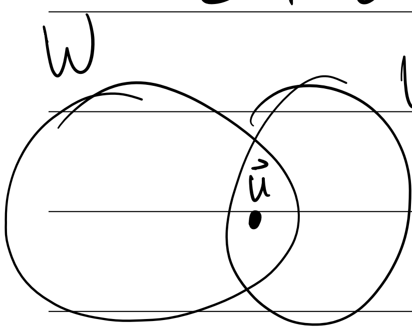
$\Rightarrow \vec{u} \cdot \vec{w} = 0$ & $\vec{v} \cdot \vec{w} = 0$.

$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} = 0 + 0 = 0$

so $\vec{u} + \vec{v} \in W^{\perp}$.

Let $k \in R$. $(k\vec{u}) \cdot \vec{w} = k(\vec{u} \cdot \vec{w}) = k(0) = 0$

so $k\vec{u} \in W^{\perp}$. $W^{\perp}$ is a subspace of R^n.



(b) Let $\vec{u} \in W \cap W^{\perp}$.

Then $\vec{u} \in W$ and $\vec{u} \in W^{\perp}$.

$\Rightarrow \vec{u} \cdot \vec{u} = 0$

(in W) (in $W^{\perp}$)

But $\vec{u} \cdot \vec{u} = 0 \Rightarrow \vec{u} = 0$ by the positivity property. Thus $W \cap W^{\perp} = \{\vec{0}\}$.

Theorem 4.9.7 If A is an $m \times n$ matrix, then:
a) The null space of A and the row space of A are orthogonal complements in R^n.
b) The null space of A^T and the column space of A are orthogonal complements in R^m.

#15 Confirm the orthogonality statements in the two parts of Theorem 4.9.7 for the matrix $A = \begin{bmatrix} 1 & 4 \\ 0 & 3 \\ -9 & 0 \end{bmatrix}$.

$\text{row}(A) = \text{span}\{[1 \;\; 0], [0 \;\; 1]\}$

$\text{null}(A) = \{\vec{0}\}$ ✓

$\text{null}(A^T) = \text{span}\left\{\begin{bmatrix} 9 \\ -12 \\ 1 \end{bmatrix}\right\} = \text{span}\{\vec{v}_1\}$

$\text{col}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}\right\} = \text{span}\{\vec{v}_2, \vec{v}_3\}$

$\vec{v}_1 \cdot \vec{v}_2 = 9 - 9 = 0$

$\vec{v}_1 \cdot \vec{v}_3 = 36 - 36 = 0$

Since the basis vectors in each set are orthogonal, the spans are orthogonal complements in R^3 (see HW for this section).

#27 Suppose that A is a 3×3 matrix whose null space is a line through the origin in 3-space. Can the row or column space of A also be a line through the origin? Explain.

Subspaces of R^3: R^3 has dimension 3

Line through origin has dim 1.

(dim [null (A)] = 1 row(A) and null(A)

are orthogonal complements dim 2 dim 1

No. It would be a plane through the origin

Theorem 4.9.8 Equivalent Statements (extends Theorem 2.3.8)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n.

k) The row vectors of A span R^n.

l) The column vectors of A form a basis for R^n.

m) The row vectors of A form a basis for R^n.

n) A has rank n.

o) A has nullity 0.

p) The orthogonal complement of the null space of A is R^n.

q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

partial proof:

b ⇒ h: $A\vec{x}$ is a linear combination of column vectors of A.
$A\vec{x} = \vec{0}$ has only the trivial solution so the column vectors are linearly independent. ✓

h ⇒ j, l, n: The n column vectors of A are linearly independent, and $\dim(\mathbb{R}^n) = n$. Thus the column vectors span $\mathbb{R}^n$ and therefore are a basis for $\mathbb{R}^n$. This also gives us that $\text{rank}(A) = n$.